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On Two Free Boundary Problems in Potential Theory

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1. INTRODUCTION

This paper deals with the following two *free boundary problems*:

Problem 1. Let $\tilde{\Omega}$ be a given bounded connected domain in the $(N-1)$-dimensional Euclidean space $\mathbb{R}^{N-1}$, $N \geqslant 2$. We then define a cylindrical domain $\Omega \subset \mathbb{R}^N$ as

$$\Omega := \left\{ x := (x_1, ..., x_N) \in \mathbb{R}^N \;\middle|\; \begin{array}{l} (x_1, ..., x_{N-1}) \in \tilde{\Omega} \\ F_0(x_1, ..., x_{N-1}) < x_N < F_1(x_1, ..., x_{N-1}) \end{array} \right\}, \tag{1.1}$$

where $F_0(x_1, ..., x_{N-1}) < F_1(x_1, ..., x_{N-1})$ are two unknown bounded continuous functions defined in $\tilde{\Omega}$. The boundary $\partial\Omega := \Gamma_0 \cup \Gamma_1 \cup \Gamma_c$ of Ω is composed of three components: the two free boundary components

$$\Gamma_i := \{x \in \mathbb{R}^N \mid x_N = F_i(x_1, ..., x_{N-1}), (x_1, ..., x_{N-1}) \in \tilde{\Omega}\}, \quad i = 0, 1, \tag{1.2}$$

and the cylindrical component

$$\Gamma_c := \{x \in \mathbb{R}^N \mid (x_1, ..., x_{N-1}) \in \partial\tilde{\Omega}, \\ F_0(x_1, ..., x_{N-1}) < x_N < F_1(x_1, ..., x_{N-1})\}. \tag{1.3}$$

Let $u(x)$ be the solution of the following boundary value problem in Ω:

$$\sum_{k=1}^{N} \frac{\partial}{\partial x_k} \left(g(|\nabla u|^2) \frac{\partial u}{\partial x_k} \right) = 0 \qquad \text{in } \Omega, \tag{1.4}$$

$$u = 0, \qquad \frac{\partial u}{\partial n} = a_0 = \text{const.} \qquad \text{on } \Gamma_0, \tag{1.5}$$

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$$u=1, \qquad \frac{\partial u}{\partial n}=a_1=\text{const.} \qquad \text{on } \Gamma_1, \tag{1.6}$$

$$\frac{\partial u}{\partial n}=0 \qquad \text{on } \Gamma_c. \tag{1.7}$$

In (1.4), g is a given positive C^2 function, assumed to satisfy the condition

$$G(\xi):=g(\xi)+2\xi g'(\xi)>0, \qquad \xi \geqslant 0, \tag{1.8}$$

that makes (1.4) uniformly elliptic. In (1.5), (1.6), and (1.7), $\partial u/\partial n$ is the outward normal derivative of u. If $g \equiv 1$, the function $u(x)$ corresponds to the electrostatic potential of a condensor. In Section 2, we prove the following result:

THEOREM 1. *The free boundary problem* (1.4), (1.5), (1.6), (1.7) *is solvable if and only if* $a_1=-a_0=:a>0$. *In this case, the free boundaries* Γ_i *are contained in two (different) horizontal hyperplanes* $F_i=c_i=\text{const.}$, $i=0$, 1, *and* $u(x)$ *depends only on* x_N.

A related free boundary problem has already been investigated by the authors in [13, 14].

Problem 2. Let Ω_k, $k=1, ..., m$ be m disconnected contractable regions in $\mathbb{R}^N$, contained in a bounded domain $\Omega_0 \subset \mathbb{R}^N$. We consider the following boundary value problem in $\Omega:=\Omega_0 \backslash \bigcup_{k=1}^{m} \bar{\Omega}_k$:

$$\sum_{k=1}^{N} \frac{\partial}{\partial x_k}\left(g(|\nabla u|^2)\frac{\partial u}{\partial x_k}\right)=-2 \qquad \text{in } \Omega, \tag{1.9}$$

$$u=0, \qquad \frac{\partial u}{\partial n}=a_0=\text{const.} \qquad \text{on } \Gamma_0:=\partial\Omega_0, \tag{1.10}$$

$$u=c_k, \qquad \frac{\partial u}{\partial n}=a_k=\text{const.} \qquad \text{on } \Gamma_k:=\partial\Omega_k,\ k=1, ..., m. \tag{1.11}$$

In (1.9), g is a given positive C^2 function, assumed to satisfy the ellipticity condition (1.8). In (1.10) and (1.11), Γ_k, $k=0, ..., m$, are $(m+1)$ free boundaries assumed to be $C^{2+\alpha}$, and $\partial u/\partial n$ is the outward normal derivative of $u(x)$. In (1.11), the values of the constants c_k are not given, but are determined from the further conditions

$$\oint_{\Gamma_k} g(|\nabla u|^2)\frac{\partial u}{\partial n}\,ds=2\,|\Omega_k|, \qquad k=1, ..., m, \tag{1.12}$$

where $|\Omega_k|:=\int_{\Omega_k} dx$ is the N-volume of Ω_k.

In two dimensions, this problem plays a role in the theory of nonlinear elasticity, where the function $u(x)$ may be interpreted as the stress function associated to an elasto-plastic beam of cross section $\Omega \subset \mathbb{R}^2$. When $g = 1$, the problem reduces to that of the elastic torsion of a hollow cylinder, in which case our result is a natural extension of Serrin's and Weinberger's results [18, 19]. The corresponding classical boundary value problem (i.e., with fixed $\partial\Omega$ but $\partial u/\partial n$ not imposed on $\partial\Omega$), has been investigated in the linear case ($g \equiv 1$) by many authors; see e.g. [7, 15]. In Section 3, we prove the following result:

THEOREM 2. *The free boundary problem* (1.9), (1.10), (1.11), *and* (1.12) *is solvable if and only if Ω is spherically symmetric; i.e., Ω_0 contains a single hole Ω_1, and Γ_0, Γ_1 are two concentric N-balls. In this case, the stress function $u(x)$ depends only on $|x| := (x_k x_k)^{1/2}$.*

The technique used in this paper was developed first by H. F. Weinberger in [19]. The main tools are best possible maximum principles together with Rellich's identity. For an account of various contributions to related free boundary problems, we refer the interested reader to the literature [1, 4, 8, 11–14, 16, 18, 19].

2. PROOF OF THEOREM 1

When appropriate, we write $u_{,i}$ instead of $\partial u/\partial x_i$, $u_{,ik}$ instead of $\partial^2 u/\partial x_i\, \partial x_k$, etc., and summation from 1 to N is assumed over each pair of repeated indices in the same monomial, unless the contrary is indicated. Moreover we use the abbreviation $q^2 := |\nabla u|^2 = u_{,k} u_{,k}$ throughout the paper.

The proof of Theorem 1 may be split into a sequence of lemmas:

LEMMA 1. *A necessary condition for the existence of a solution $u(x)$ of Problem 1 is*

$$a_1 = -a_0 =: a > 0. \tag{2.1}$$

Using the divergence theorem and the differential equation (1.4), we have

$$\oint_{\partial\Omega} u_{x_N} g(q^2) \frac{\partial u}{\partial n}\, ds = \int_\Omega g(q^2)\, u_{x_N, i} u_{,i}\, dx. \tag{2.2}$$

We observe that the right-hand side in (2.2) may also be expressed as a boundary integral since we have

$$g(q^2)\, u_{x_N, i} u_{,i} = \frac{1}{2} \frac{\partial}{\partial x_N} \left\{ \int_0^{q^2} g(\xi)\, d\xi \right\}. \tag{2.3}$$

Another application of the divergence theorem leads then to

$$\int_\Omega g(q^2)\, u_{x_N, i} u_{,i}\, dx = \frac{1}{2} \oint_{\partial\Omega} \left\{ \int_0^{q^2} g(\xi)\, d\xi \right\} n_N\, ds, \tag{2.4}$$

where $(n_1, ..., n_N)$ is the exterior normal unit vector on $\partial\Omega$. Combining (2.2) and (2.4), taking (1.5), (1.6), and (1.7) into account, and noting that

$$n_N = 0 \qquad \text{on} \quad \Gamma_c, \tag{2.5}$$

we obtain

$$\int_0^{a_0^2} G(\xi)\, d\xi \oint_{\Gamma_0} n_N\, ds + \int_0^{a_1^2} G(\xi)\, d\xi \oint_{\Gamma_1} n_N\, ds = 0, \tag{2.6}$$

where $G(\xi)$ is defined in (1.8). Moreover, we have in view of (2.5)

$$0 = \oint_{\partial\Omega} n_N\, ds = \oint_{\Gamma_0} n_N\, ds + \oint_{\Gamma_1} n_N\, ds. \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$\oint_{\Gamma_1} n_N\, ds \int_{a_0^2}^{a_1^2} G(\xi)\, d\xi = 0. \tag{2.8}$$

Since the first integral in (2.8) is strictly positive, we conclude, using (1.8), that $a_0^2 = a_1^2$, i.e., Lemma 1 is proved.

Now we establish a modified version of Rellich's identity formulated in the next

LEMMA 2. *We have*

$$|\Omega| \int_0^{a^2} G(\xi)\, d\xi = \int_\Omega \left\{ \int_0^{q^2} G(\xi)\, d\xi - 2g(q^2) \sum_{k=1}^{N-1} u_{x_k}^2 \right\} dx, \tag{2.9}$$

where $|\Omega|$ *is the N-volume of* Ω, $a = |\partial u/\partial n| = \text{const.}$ *on* $\Gamma_0 \cup \Gamma_1$.

For the proof of Lemma 2, we compute, using the differential equation (1.4)

$$(x_N u_{x_N} g(q^2)\, u_{,i})_{,i} = u_{x_N}^2 g(q^2) + x_N u_{x_N,i} u_{,i}\, g(q^2)$$
$$= u_{x_N}^2 g(q^2) + \frac{1}{2}\frac{\partial}{\partial x_N}\left(x_N \int_0^{q^2} g(\xi)\, d\xi\right) - \frac{1}{2}\int_0^{q^2} g(\xi)\, d\xi. \tag{2.10}$$

Integrating (2.10) over Ω and making use of the divergence theorem, we obtain

$$\oint_{\partial\Omega} x_N u_{x_N} g(q^2)\frac{\partial u}{\partial n}\, ds = \int_\Omega u_{x_N}^2 g(q^2)\, dx + \frac{1}{2}\oint_{\partial\Omega}\left\{\int_0^{q^2} g(\xi)\, d\xi\right\} x_N n_N\, ds$$
$$-\frac{1}{2}\int_\Omega\left\{\int_0^{q^2} g(\xi)\, d\xi\right\} dx. \tag{2.11}$$

We note that the two boundary integrals in (2.11) are nonzero on $\Gamma_0 \cup \Gamma_1$ only, in view of (1.7) and (2.5). Moreover, we have $(\partial u/\partial n)^2 = a^2 =$ const. on $\Gamma_0 \cup \Gamma_1$, as a consequence of Lemma 1. Therefore the identity (2.11) may be rewritten as

$$\int_0^{a^2} G(\xi)\, d\xi \oint_{\Gamma_0\cup\Gamma_1} x_N n_N\, ds = \int_\Omega \left\{\int_0^{q^2} G(\xi)\, d\xi - 2g(q^2)\sum_{k=1}^{N-1} u_{x_k}^2\right\} dx. \tag{2.12}$$

Identity (2.9) follows since we have

$$\oint_{\Gamma_0\cup\Gamma_1} x_N n_N\, ds = \oint_{\partial\Omega} x_N n_N\, ds = |\Omega|. \tag{2.13}$$

The last step in the proof of Theorem 1 is the following maximum principle:

LEMMA 3. *The function q^2 defined in $\bar{\Omega}$ on solutions of* (1.4), *takes its maximum value at an interior point of Ω if and only if $q^2 \equiv$* const. *in Ω.*

The proof of Lemma 3 is based on the construction of an appropriate elliptic differential inequality for q^2. The conclusion of Lemma 3 is then a direct consequence of Hopf's first maximum principle [5, 17]. The reader is referred to [14] or to [16] for computational details.

We note that $\partial q^2/\partial n = 0$ on Γ_c in view of (1.7). Hopf's second maximum principle [6, 17] implies then that q^2 can take its maximum value on Γ_c if and only if $q^2 \equiv$ const. in Ω, so that we have

$$q^2 \leqslant a^2 \qquad \text{in } \bar{\Omega}. \tag{2.14}$$

Let us return to the proof of Theorem 1. Using (2.14) and (2.9), we obtain

$$\int_\Omega \left\{\int_0^{q^2} G(\xi)\,d\xi\right\} dx \leqslant \int_\Omega \left\{\int_0^{q^2} G(\xi)\,d\xi - 2g(q^2) \sum_{k=1}^{N-1} u_{x_k}^2\right\} dx. \tag{2.15}$$

As an immediate consequence of (2.15), we have

$$\int_\Omega \sum_{k=1}^{N-1} u_{x_k}^2\,dx = 0, \tag{2.16}$$

i.e.,

$$u_{x_k} \equiv 0 \qquad \text{in} \quad \Omega,\ k = 1, ..., N-1. \tag{2.17}$$

This establishes the desired result: $u(x) = u(x_N)$.

We remark that the free boundaries Γ_0 and Γ_1 need not be representable as indicated in (1.2), i.e., as graphs of functions $F_i(x_1, ..., x_{N-1})$, $i = 0, 1$. This has been assumed only to make the formulation of Problem 1 easier.

To conclude this section, we modify the free boundary problem 1 by taking $F_0 = 0$; i.e., $\Gamma_0 := \{(x_1, ..., x_{N-1}, 0) \in \mathbb{R}^N,\ (x_1, ..., x_{N-1}) \in \tilde{\Omega}\}$, but $\partial u/\partial n$ is not prescribed any more on Γ_0. It can be shown that this modified version of Problem 1 is solvable if and only if the free boundary Γ_1 is contained in a horizontal hyperplane $F_1 = c_1 = \text{const.}$, and $u(x)$ depends only on x_N. The proof of this result is similar to the proof of Theorem 1 and is omitted.

3. Proof of Theorem 2

In this section, we establish the result stated in Theorem 2. The various steps of the proof are again presented as a sequence of lemmas.

Lemma 4. *Let a_k be the normal derivative of u on Γ_k, $k = 0, ..., m$, where $u(x)$ is the stress function of Problem 2 defined in* (1.9), (1.10), (1.11), (1.12). *We have*

$$a_0\, g(a_0^2) = -2\,\frac{|\Omega_0|}{|\Gamma_0|} < 0, \tag{3.1}$$

$$a_k\, g(a_k^2) = 2\,\frac{|\Omega_k|}{|\Gamma_k|} > 0, \qquad k = 1, ..., m. \tag{3.2}$$

The relationships (3.2) are an immediate consequence of conditions (1.12) together with the boundary conditions (1.11). The proof of (3.1) follows from the divergence theorem. Integrating (1.9) over Ω, we have

$$\oint_{\partial\Omega} g(q^2)\frac{\partial u}{\partial n}\,ds = \int_\Omega (g(q^2)\,u_{,i})_{,i}\,dx = -2\int_\Omega dx = -2\left\{|\Omega_0| - \sum_{k=1}^m |\Omega_k|\right\}, \tag{3.3}$$

or, with the boundary conditions (1.10), (1.11),

$$a_0\,g(a_0^2)\,|\Gamma_0| + \sum_{k=1}^m a_k\,g(a_k^2)\,|\Gamma_k| = -2\left\{|\Omega_0| - \sum_{k=1}^m |\Omega_k|\right\}. \tag{3.4}$$

A combination of (3.2) and (3.4) leads to (3.1).

LEMMA 5. *The function $\Phi(x)$ defined in $\bar{\Omega}$ on solutions of* (1.9) *as*

$$\Phi(x) := N\int_0^{q^2} G(\xi)\,d\xi + 4u, \tag{3.5}$$

takes its maximum value at an interior point of Ω if and only if $\Phi \equiv$ const. *in Ω.*

The proof of Lemma 5 is based on the construction of an appropriate elliptic differential inequality for the function $\Phi(x)$. In fact, we have

$$\Delta\Phi + \frac{2g'}{g}\,\Phi_{,ij}u_{,i}u_{,j} + W_k\Phi_{,k} = \chi_{ij}\chi_{ij} \geq 0 \qquad \text{in } \Omega, \tag{3.6}$$

with

$$\chi_{ij} := u_{,ij} + \frac{2g'}{g}\,u_{,ik}u_{,k}u_{,j} + \frac{2}{Ng(q^2)}\,\delta_{ij}. \tag{3.7}$$

In (3.6), W_k is a vector field regular throughout Ω, and in (3.7), δ_{ij} is the Kronecker symbol. For computational details, we refer the reader to [9, 10]. The conclusion of Lemma 5 is now a direct consequence of Hopf's first principle [5, 17].

We prove now the following version of Rellich's identity:

LEMMA 6. *We have*

$$\int_\Omega \Phi\,dx = |\Omega_0|\,\Phi(\Gamma_0) - \sum_{k=1}^m |\Omega_k|\,\Phi(\Gamma_k). \tag{3.8}$$

For the proof of (3.8), we write

$$0 = x_i u_{,i} \{(g(q^2) u_{,j})_{,j} + 2\} = (x_i u_{,i} g(q^2) u_{,j})_{,j} - q^2 g(q^2) + \frac{N}{2} \int_0^{q^2} g(\xi)\, d\xi - \frac{1}{2} \frac{\partial}{\partial x_i} \left(x_i \int_0^{q^2} g(\xi)\, d\xi \right) + 2(x_i u)_{,i} - 2Nu. \quad (3.9)$$

Integrating (3.9) over Ω and making use of the divergence theorem, we obtain in view of (1.10), (1.11)

$$\int_\Omega q^2 g(q^2)\, dx + 2N \int_\Omega u\, dx - \frac{N}{2} \int_\Omega \left\{ \int_0^{q^2} g(\xi)\, d\xi \right\} dx = \oint_{\partial\Omega} x_i n_i \left[q^2 g(q^2) - \frac{1}{2} \int_0^{q^2} g(\xi)\, d\xi \right] ds + 2 \oint_{\partial\Omega} u x_i n_i\, ds. \quad (3.10)$$

The first two terms in (3.10) may be transformed using the identity

$$\oint_{\partial\Omega} g(q^2)\, u \frac{\partial u}{\partial n}\, ds = \int_\Omega q^2 g(q^2)\, dx - 2 \int_\Omega u\, dx. \quad (3.11)$$

This leads to

$$\int_\Omega \Phi(x)\, dx = \oint_{\partial\Omega} \left\{ \int_0^{q^2} G(\xi)\, d\xi \right\} x_i n_i\, ds + 4 \oint_{\partial\Omega} x_i n_i u\, ds + 2(N-1) \oint_{\partial\Omega} u \frac{\partial u}{\partial n} g(q^2)\, ds, \quad (3.12)$$

where $\Phi(x)$ and $G(q^2)$ are defined in (3.5) and in (1.8). Taking the boundary conditions (1.10), (1.11) into account, and making use of (3.1), (3.2), we obtain the desired result (3.8).

The next step establishes a comparison between the values of Φ on the different boundary components Γ_k, $k = 0, ..., m$:

LEMMA 7. *We have*

$$\Phi(\Gamma_0) \leqslant \max_{k \in \{1, ..., m\}} \Phi(\Gamma_k). \quad (3.13)$$

For the proof of Lemma 7, we start from the converse hypothesis:

$$\Phi(\Gamma_0) > \max_{k \in \{1, ..., m\}} \Phi(\Gamma_k), \quad (3.14)$$

and show that this hypothesis leads to a contradiction. Indeed, inequality (3.14) and Lemma 5 lead to

$$\Phi(x) \leqslant \Phi(\Gamma_0), \qquad \forall x \in \Omega. \tag{3.15}$$

Integrating (3.15) over Ω and using identity (3.8) leads to

$$|\Omega_0|\, \Phi(\Gamma_0) - \sum_{k=1}^{m} |\Omega_k|\, \Phi(\Gamma_k) = \int_\Omega \Phi(x)\, dx \leqslant \Phi(\Gamma_0) \left\{ |\Omega_0| - \sum_{k=1}^{m} |\Omega_k| \right\}, \tag{3.16}$$

from which we obtain

$$\left(\max_{k \in \{1, \ldots, m\}} \Phi(\Gamma_k) \right) \sum_{k=1}^{m} |\Omega_k| \geqslant \sum_{k=1}^{m} |\Omega_k|\, \Phi(\Gamma_k) \geqslant \Phi(\Gamma_0) \sum_{k=1}^{m} |\Omega_k|, \tag{3.17}$$

or

$$\Phi(\Gamma_0) \leqslant \max_{k \in \{1, \ldots, m\}} \Phi(\Gamma_k). \tag{3.18}$$

Inequality (3.18) contradicts (3.14).

We now complete the proof of Theorem 2. According to Lemmas 5 and 7, the auxiliary function $\Phi(x)$ takes its maximum value on some interior boundary component, say Γ_1. This implies that

$$\frac{\partial \Phi}{\partial n} = 2NG(q^2)\, u_n u_{nn} + 4u_n \geqslant 0 \tag{3.19}$$

at each point of Γ_1. Using the differential equation (1.9) rewritten in normal coordinates as

$$u_{nn} G(q^2) - (N-1)\, K u_n\, g(q^2) + 2 = 0 \qquad \text{on } \Gamma_1, \tag{3.20}$$

where K is the average curvature of Γ_1, we can eliminate the second normal derivative of u in (3.19). This leads to

$$\frac{\partial \Phi}{\partial n} = 2(N-1)\, a_1 \{ NKa_1\, g(a_1^2) - 2 \} \geqslant 0 \qquad \text{on } \Gamma_1. \tag{3.21}$$

According to (3.2), a_1 must be a positive constant. We thus obtain

$$NKa_1\, g(a_1^2) - 2 \geqslant 0 \qquad \text{on } \Gamma_1. \tag{3.22}$$

Integrating (3.22) over Γ_1 and using (3.2), we are led to the geometric inequality

$$N\, |\Omega_1| \oint_{\Gamma_1} K\, ds - |\Gamma_1|^2 \geqslant 0. \tag{3.23}$$

Inequality (3.23) coincides with one of Minkowski's isoperimetric inequalities between mixed volumes [2, 3], except for the inequality sign that is reversed. From this analysis, we conclude that we must have *equality* in (3.23), so that Ω_1 must be an N-ball. Moreover, we have equality in (3.21). Hopf's second principle [6, 17] implies then that $\Phi(x)$ is identically constant in Ω. This shows that we must have equality in (3.6), i.e.,

$$\chi_{ij} := u_{,ij} + \frac{2g'}{g} u_{,ik} u_{,k} u_{,j} + \frac{2}{Ng(q^2)} \delta_{ij} \equiv 0 \qquad \text{in } \Omega. \tag{3.24}$$

Equation (3.6) with the equality sign and (3.24) imply

$$(g(q^2) u_{,i})_{,j} = -\frac{2}{N} \delta_{ij} \qquad \text{in } \Omega. \tag{3.25}$$

An integration of (3.25) leads to

$$g(q^2) u_{,i} = -\frac{2}{N} x_i \qquad \text{in } \Omega \tag{3.26}$$

for a suitable choice of the origin. Equation (3.26) shows that q^2 depends only on $|x| := (x_i x_i)^{1/2}$, so that ∇u is a central field. The level sets $\{u = \text{const.}\}$ are therefore spherically symmetric, implying that there is no more than one single interior boundary component Γ_1, and that Γ_0 is an N-ball concentric to Γ_1. This achieves the proof of Theorem 2.

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